DESIGN OF A LAYERED THERMALLY STABLE SPHERE OF MINIMAL THICKNESS MADE FROM A FINITE SET OF MATERIALS

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Thickness optimization is considered for a multilayer spherical shell that damps the amplitude of the internal temperature fluctuations.

In [1, 2], there are discussions of optimizing the mass (thickness) of planar layered heat-shield panels to damp out external temperature perturbations acting on one surface to a given level. A major feature was that the panel can be made of a finite set of materials. The number, dimensions, and materials in the layers are determined by solving the corresponding optimal-control problem. Here we consider the spherically symmetrical case. The optimization is formulated as follows.

Using a finite set of materials, we have to synthesize a layered sphere of minimal thickness that will damp the amplitude of the thermal oscillations $T_0(\tau)=\mathrm{Re}[A_0\exp(i\omega\tau)]$ acting on the inner surface by a given factor. Here $\omega=2\pi/\tau^*$ and A_0 is the complex amplitude. The internal radius R_0 is specified.

The controls are provided by the thermal conductivity distribution over the radius $\lambda(r)$ and the outside radius R. A knowledge of $\lambda(r)$ enables one to determine simultaneously the number, thicknesses, and materials in the layers.

Because the set of materials is discrete, $\lambda(r)$ is a piecewise-constant function, whose value range belongs to a finite discrete set:

$$\lambda(r)=\{\lambda_s: r_s\leqslant r<r_{s+1},\ s=1,\ \dots,\ S\}, \tag{1}$$

$$\lambda_s\in\Lambda=\{\Lambda^1,\ \Lambda^2,\ \dots,\ \Lambda^k\}. \tag{2}$$

Set Λ consists of the thermal conductivities of the given set of materials.

In the periodic steady state, one utilizes the linearity to represent the temperature pattern within the layered sphere as

$$T(r,\ \tau)=\mathrm{Re}[A(r)\exp(i\omega\tau)]. \tag{3}$$

We introduce the continuous functions $y_1(r) = A(r)$ and $y_2(r) = \lambda(r)A'(r)$, which are the complex amplitudes of the temperature and heat flux. The thermal-conduction equation is used with ideal contact between the layers to derive the following equations for y_1 and y_2:

$$y_1'=\frac{1}{\lambda}y_2\equiv f_1,\quad y_2'=\frac{i\omega\lambda}{a(\lambda)}y_1-\frac{2}{r}y_2\equiv f_2. \tag{4}$$

The boundary conditions are as follows:

$$y_2(R_0)=\alpha_1[y_1(R_0)-A_0],\quad y_2(R)=\alpha_2y_1(R). \tag{5}$$

System (4) is given throughout the interval $[R_0, R]$; by virtue of (1) and (2), the right sides in (4) are piecewise continuous. We describe the functionals appearing in the optimization formulation. The functional to be minimized is

$$F_0[\lambda(r),\ R]\equiv\int\limits_{R_0}^{R}dr=R-R_0. \tag{6}$$

The constraint imposed on the amplitude of $T(R, \tau)$ is put as

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TABLE 1. Thermophysical Properties of the Initial Material Set

Material number	Material	λ, W/(m · K)	$a \cdot 10^6$, m²/sec	ρ, kg/m²
	Plates:			
1	mineral wool	0,0516	0,417	200
2	mineral wool	0,0671	0,309	350
3	wood fiber	0,0774	0,076	600
4	Foamed concrete	0,1032	0,278	600
5	Arbolite	0,1204	0,088	800
6	Foamed concrete	0,1548	0,311	800
7	Foamed concrete	0,215	0,347	1000
8	Plywood	0,086	0,084	600

$$F_1[\lambda(r),\ R] \equiv y_1(R)\,\overline{y_1}(R) - \eta^2 |A|_0^2 = 0. \tag{7}$$

Here η is a factor defining the degree of damping for the amplitude of the external perturbations, while the overbar denotes the complex conjugate.

The minimization is formulated as follows. Among the piecewise-constant function λ(r) defined by (1) and (2) and the numbers $R \in [R_0, \infty)$ we have to find a pair $(\lambda^{opt}(r), R^{opt})$ that minimizes the functional of (6) subject to (7). The phase variable y_1 in (7) satisfies the boundary-value problem of (4) and (5).

The main feature of this problem is that the region for the values of the control function in (2) is discrete, which does not allow us to follow the usual approach of constructing small variations in the uniform norm:

$$\|\delta\lambda\| = \max_{r\in[R_0,\ R]} |\delta\lambda(r)|.$$

It is therefore important to use needle-type variations [3] in deriving the necessary optimality conditions and in constructing the algorithm. By variation in the control function λ(r) we understand a functional λ*(r) taking the form

$$\lambda^*(r) = \begin{cases} \vartheta; & r \in M,\ \vartheta \in \Lambda, \\ \lambda(r); & r \notin M, \end{cases}$$

where $M \subset [R_0, R]$ is a set whose measure is small. A variation in the control $\{\lambda^*(r), \delta R\}$ generates the following variations in the functionals δF_0 and δF_1 [4]

$$\delta F_0[\lambda^*(r),\ \delta R] = \delta R \tag{8}$$

and

$$\delta F_1[\lambda^*(r),\ \delta R] = \int_M \mathrm{Re} < \psi,\ f(\vartheta) - f(\lambda) > dr + 2\,\mathrm{Re}\left[\overline{y_1}\left(\frac{y_2}{\lambda} + \varphi_1\right)\right]_{r=R} \delta R.$$

Here $f = \{f_1, f_2\}$; $<\cdot,\cdot>$ is the scalar product. The conjugate variables $\psi = \{\psi_1, \psi_2\}$ and $\varphi = \{\varphi_1, \varphi_2\}$ satisfy the boundary-value problems

$$\psi_1' = \frac{1}{\lambda}\psi_2 + \frac{2}{r}\psi_1, \quad \psi_2' = \frac{i\omega\lambda}{a(\lambda)}\psi_1,$$
$$\psi_2(R_0) = \alpha_1\psi_1(R_0), \quad \psi_2(R) = \alpha_2\psi(R) + 2\overline{y_1}(R) \tag{9}$$

$$\varphi_1' = \frac{1}{\lambda}\varphi_2, \quad \varphi_2' = \frac{i\omega\lambda}{a(\lambda)}\varphi_1 - \frac{2}{r}\varphi_2,$$
$$\varphi_2(R_0) = \alpha_1\varphi_1(R_0),$$
$$\varphi_2(R) = \alpha_2\left[\varphi_1(R) + \frac{1}{\lambda}y_2(R)\right] - \frac{i\omega\lambda}{a(\lambda)}y_1(R) - \frac{2}{r}y_2(R). \tag{10}$$

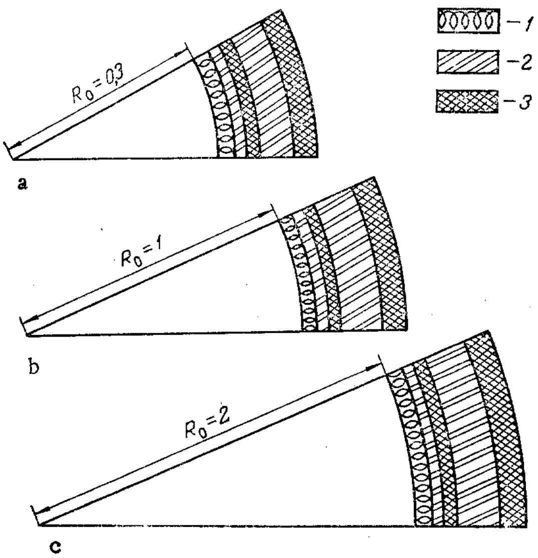

Fig. 1. Examples of spherical shells optimal in thickness: 1) material 1 from Table 2; 2) material 3; 3) material 5; R_0, m.

As the constraint $\delta F_1 = 0$ applies to the perturbed control, the variation in the minimized functional can finally be put as

$$\delta F_0[\lambda^*, \delta R] = \int_M [H(y, \psi, \varphi, \lambda) - H(y, \psi, \varphi, \vartheta)]\, dr, \tag{11}$$

where

$$H(y, \psi, \varphi, \vartheta) = \mathrm{Re}\left[\frac{y_2\psi_2}{\vartheta} - \frac{i\vartheta\omega}{a(\vartheta)} y_1\psi_1\right]\left[2\frac{\alpha_2\eta^2}{\lambda(R)} + \bar{y}_1\varphi_1\right]^{-1}_{r=R}. \tag{12}$$

For the control $\{\lambda(r), R\}$ to be optimal, it is necessary to obey the condition $\delta F_0 \geqslant 0$ for all permissible variations. As the set with small measure M can be chosen everywhere compact in the interval $r \in [R_0, R]$, the latter condition is equivalent to the following:

$$H(y, \psi, \varphi, \lambda^{opt}) (=) \max_{\vartheta \in \Lambda} H(y, \psi, \varphi, \vartheta), \tag{13}$$

which applies for almost all r. The necessary optimality conditions can be formulated as follows. Let $\{\lambda^{opt}(r), R^{opt}\}$ be the optimal control that minimizes (6), while $y = \{y_1, y_2\}$ is the corresponding phase locus. Then there exist vector functions $\psi = \{\psi_1, \psi_2\}$ and $\varphi = \{\varphi_1, \varphi_2\}$, defined from (9) and (10) and such that the Hamilton function of (12) constructed with them attains its maximum value with respect to the argument ϑ on the optimum control for almost all $r \in [R_0, R]$.

Expression (11) for the increment in F_0 is used to construct a minimizing control sequence [4]. Each step consists in sequential improvement in the control on the small-measure sets M_i, each of which consists of sufficiently small segments into which $[R_0, R]$ is divided. If λ^{opt} takes the same value on two or more successive segments, these are all combined into one homogeneous layer.

We consider the following problem as an example. The materials given in Table 1 are to be used in designing a layered sphere of minimal thickness to damp temperature-oscillation amplitudes by a factor 25 ($\eta = 0.04$). The coefficients α_1 and α_2 in the boundary conditions of (5) are correspondingly 23.2 and 0 ($W/m^2 \cdot K$). As the small-measure set we sequentially take small segments of length 0.25×10^{-2} m filling the segment $[R_0, R]$.

Figure 1 shows optimal shells for $R_0 = 0.3$, 1, and 2 m. The shell becomes thicker as the internal radius increases because of increases in the second and penultimate layers. The optimum sphere consists of five layers. Out of the given set of materials, the optimum design contains the first, third, and fifth. Total shell thicknesses 14.5, 15.75, and 16 cm correspondingly for $R_0 = 0.3$, 1, and 2 m.

One can compare these spherical shells with analogous planar layered panels designed for the same conditions [4], which shows that the thickness and structure of a spherical shell tend to those for a planar panel as the radius increases; the two become virtually identical for $R_0 > 2$ m.

NOTATION

r, current radius; R_0, inside radius of sphere; R, outside radius; λ, a, ρ, thermal conductivity, thermal diffusivity, and density, respectively; T, temperature; τ, time; τ^*, period; ω, frequency of temperature fluctuations; α_1, α_2, heat-transfer coefficients.

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AN INVERSE STEFAN'S PROBLEM IN CASTING MULTICOMPONENT ALLOYS

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A model is considered that incorporates the feature that melting (crystallization) occurs over a certain temperature range, and it is shown that the solution to the internal inverse problem is unique in one such formulation.

Determining the temperature pattern in a casting during crystallization is a nonlinear problem of the free-boundary class, in which part of the boundary is unspecified and must be determined when the differential equations are solved by the use of an additional boundary condition at that part. This relationship is readily derived from the heat-balance equation and is called Stefan's condition, which in the one-dimensional case takes the form

$$\left[\lambda_1 \frac{\partial}{\partial x} T_1(x,\ t) - \lambda_2 \frac{\partial}{\partial x} T_2(x,\ t)\right]\Bigg|_{x=s(t)} = r\gamma s'(t).$$

There are many papers on this topic, but in them it is either assumed that the crystallization occurs at a fixed temperature rather than over a certain range or that the treatment can be reduced to that.

We consider a schematic model for the phase-transition zone in casting a multicomponent alloy. During melting (cooling), a transitional layer is formed, which may be considered as a thermally active resistance.

Let the transitional layer have thickness $\delta > 0$, which may be fairly small. We denote by $R_0(x)$ the thermal-resistance density in the transitional layer, $R_0(x) = R'(x)$, while $r_0(x)$ denotes the density of the phase-transition latent heat, $r_0(x) = r'(x)$; we now apply Kirchoff's and Ohm's laws to the part [x, x + dx] of the transition layer to get

$$\mathrm{dI}(x,\ t) = \gamma s'(t)\, r_0(x)\, \mathrm{d}x, \quad \mathrm{dT}(x,\ t) = I(x,\ t)\, R_0(x)\ \mathrm{d}x,$$

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